

Displacement, Velocity and Acceleration in one dimension

In this document we consider the general relationship between displacement, velocity and acceleration. Displacement, velocity and acceleration are all considered to be vector quantities; they have direction as well as magnitude. For simplicity the concepts are considered in one dimension or along a straight line in which case displacement, velocity or acceleration can each be represented as a scalar function of time. The concepts can be extended straightforwardly to two or three dimensions, in which cases displacement, velocity and acceleration can be represented as vector functions of time.

Displacement, velocity and acceleration are generally related to each other mathematically through differentiation¹ and integration². The relationships follow straightforward formulae in the special cases of constant velocity³ or constant acceleration⁴. The relationship between displacement, velocity and acceleration is often termed *kinematics*.

Displacement

The displacement of an object is a measure of its position relative to some reference point, often its initial position. The displacement of an object is therefore a straightforward function of time: $x(t)$, or x . The SI unit for displacement is the metre.

Velocity in terms of displacement

Velocity is the rate of change of displacement with respect to time and it is often denoted

$$\dot{x}(t) = \frac{dx}{dt}. \tag{1}$$

Hence on a displacement-time graph the velocity can be found by evaluating the gradient. The standard unit of velocity is metre per second (ms^{-1}).

Example

The displacement of an object (in metres) follows the relationship

$$x(t) = \frac{t^3}{6} + t.$$

Find an expression for the velocity of the object. State the velocity of the object at $t = 10\text{s}$.

¹ [Differentiation](#)

² [Integration](#)

³ [Distance travelled under constant Velocity](#)

⁴ [Distance travelled under constant Acceleration](#)

Answer

Differentiating, the velocity of the object can be expressed as follows:

$$\dot{x}(t) = \frac{t^2}{2} + 1.$$

At $t = 10\text{s}$ the velocity of the object is given by

$$\dot{x}(10) = \frac{10^2}{2} + 1 = 51.$$

Hence the velocity at $t = 10\text{s}$ is 51ms^{-1} .

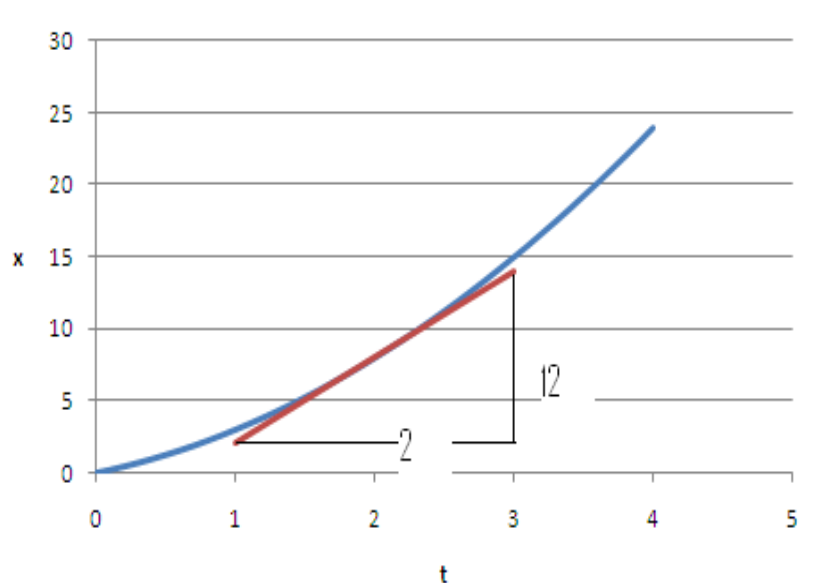
Velocity can be determined manually from a displacement-time graph by finding the gradient.

Example

The curved line in the following graph represents an object's displacement with respect to time. Determine the velocity at $t = 2\text{s}$.

Answer

Drawing a tangent to the line we can see that the gradient = $12/2$. Hence the velocity is 6ms^{-1} .



Displacement in terms of velocity

Since integration is the inverse of differentiation, equation (1) may be re-written as follows:

$$x(t) = \int_0^t \dot{x}(t) dt,$$

(2)

where the displacement at time $t = 0$ is taken to be zero.

Example

The velocity of an object (in ms^{-1}) follows the relationship

$$\dot{x}(t) = e^{2t}$$

Find a general expression for the displacement of the object. State the displacement of the object after 3s.

Answer

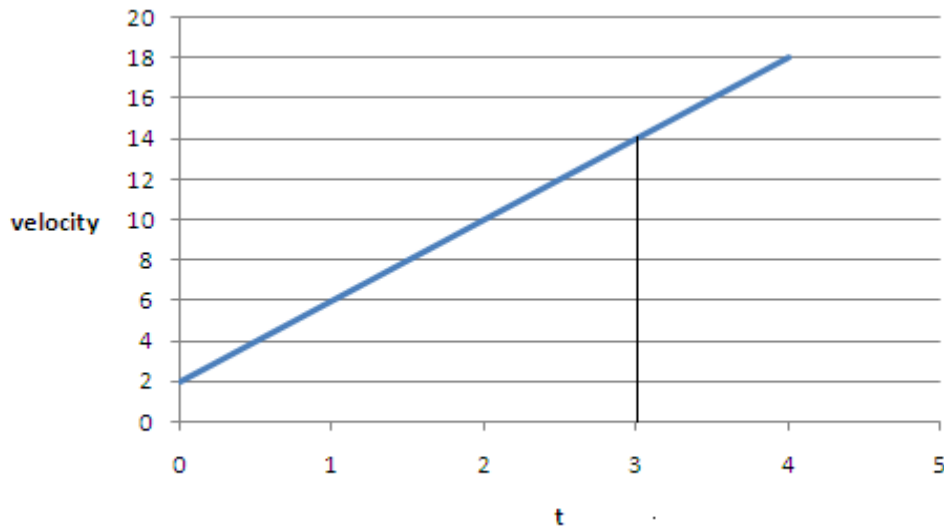
$$x(t) = \int_0^t \dot{x}(t) dt = \int_0^t e^{2t} dt = \left[\frac{1}{2} e^{2t} \right]_0^t = \frac{1}{2} e^{2t} - \frac{1}{2}$$

$$\text{At } t = 3\text{s}, x(3) = \frac{1}{2} e^{2 \times 3} - \frac{1}{2} = \frac{1}{2} e^6 - \frac{1}{2} = 201.2144 \text{ m.}$$

Since the integral represents the area under the graph, then the displacement can be determined by finding the area underneath a velocity-time graph

Example

The following graph represents an objects velocity plotted against time. Determine the distance travelled after 3s.



Answer

The area under the graph between 0 and 3s has the shape of a trapezium.

The area of the trapezium is 24 so the distance travelled is 24m.

Acceleration in terms of velocity and displacement

Acceleration is the rate of change of velocity with respect to time. It is the second derivative of the displacement;

$$\ddot{x}(t) = \frac{d\dot{x}}{dt} = \frac{d^2x}{dt^2}.$$

(3)

The standard unit for acceleration is metres per second squared ms^{-2} .

Example

The velocity of an object (in ms^{-1}) follows the relationship

$$\dot{x}(t) = 2\sin(5t)$$

Find a general expression for the acceleration of the object. State the acceleration of the object at $t = 3\text{s}$.

Answer

The general expression for

$$\ddot{x}(t) = \frac{d\dot{x}}{dt} = 10\cos(5t)$$

At $t = 3\text{s}$ the acceleration is $\ddot{x}(3) = \left. \frac{d\dot{x}}{dt} \right|_{t=3} = 10\cos(5 \times 3) = 10\cos(15) = 9.859$.

Hence the acceleration after 3s is 9.859ms^{-2} .

Velocity in terms of acceleration

Since integration is the inverse of differentiation, equation (2) may be re-written as follows:

$$\dot{x}(t) = \int_0^t \ddot{x}(t) dt,$$

(3)

where the velocity at time $t = 0$ is taken to be zero.

Exercise

The acceleration of an object is given by the expression:

$$\ddot{x}(t) = 3 \sin\left(2t + \frac{\pi}{4}\right).$$

Determine an expression for the velocity. State the velocity at $t=3$ s.

Answer

Integrating the expression above, we obtain the formula for the acceleration:

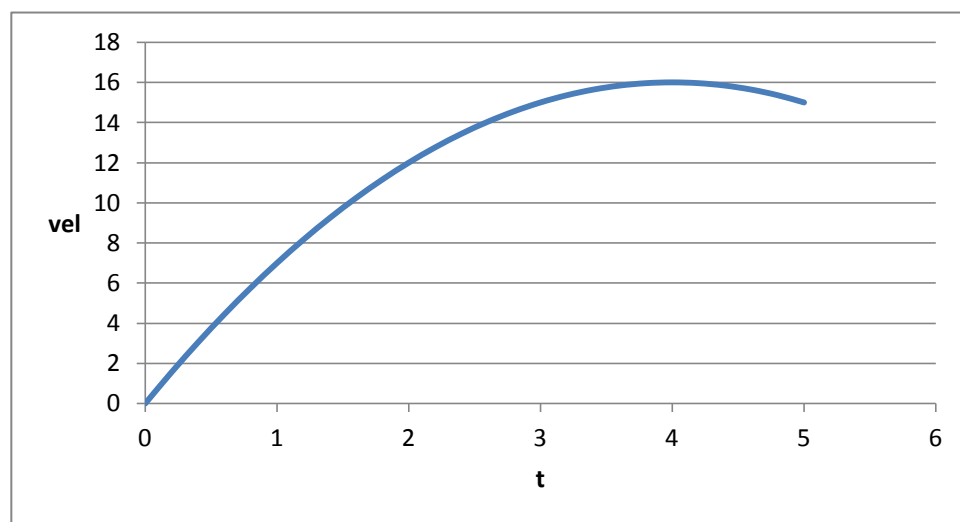
$$\dot{x}(t) = -1.5 \cos\left(2t + \frac{\pi}{4}\right).$$

When $t=3$, the acceleration is given by $\dot{x}(3) = -1.5 \cos\left(2 \times 3 + \frac{\pi}{4}\right) = -1.3148 \text{ ms}^{-2}$.

Acceleration can be determined by finding the gradient from a velocity-time graph.

Exercise

The velocity of an object is shown in the following graph.



Estimate the acceleration at $t=3$ s and the displacement after 2s.

By finding the gradient at $t=3$ the estimate for the acceleration is 2ms^{-2} .

By finding the area under the graph between 0 and 2s the estimate for the displacement is 12m.